LISA observations of supermassive black holes using the full post-Newtonian inspiral waveforms

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Outline

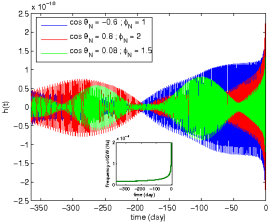
1. Introduction
   - Coalescence of SMBH binaries
   - Effect of higher harmonics on the SNR
2. Methodology
   - Fisher information matrix formalism
   - Stationary Phase Approximation
   - Working hypotheses
3. Results
   - The impact of the FWF: General trends
   - Exploring the parameter space
   - Sirens: Pre-merger localization

Inspiral waveforms. PN approximation

- ☞ We are studying the inspiral stage of SMBH binaries, which waveform can be described analytically (PN approximation).
- ☞ Our waveform is characterized by 11 independent parameters:

$$\{\cos\theta_N, \phi_N, \cos\theta_L, \phi_L, \ln D_L, t_c, \phi_c, \beta, \sigma\} + \left\{ \begin{matrix} (\delta m \ , \ M) \\ (\mathcal{M} \ , \ \mu) \end{matrix} \right\}$$

$$\left[\mu = \frac{m_1 \cdot m_2}{m_1 + m_2} \qquad \mathcal{M} = \mu^{3/5} M^{2/5} \right]$$



Inspiral waveform up to 2PN

$$h^{(\mathrm{FWF})}_{+,\times} = \frac{2M\eta}{D_L}(M\omega)^{2/3}\left\{ H^{(0)}_{+,\times} + v^{1/2} H^{(1/2)}_{+,\times} + v\, H^{(1)}_{+,\times} + v^{3/2} H^{(3/2)}_{+,\times} + v^2 H^{(2)}_{+,\times} \right\}$$

where we are working with *geometric coordinates*, $v \equiv (M\omega)^{2/3}$ and

$$H^{(m/2)}_{+,\times} = \sum_j \left[A^{(m/2)}_{j,(+,\times)} \cos(\frac{j}{2}\Phi + \varphi_D) + B^{(m/2)}_{j,(+,\times)} \sin(\frac{j}{2}\Phi + \varphi_D) \right]$$

Here, Φ is the phase of the second harmonic, which is also written as a PN expansion.

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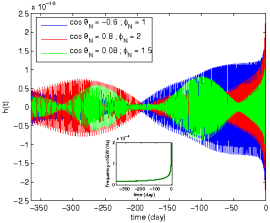

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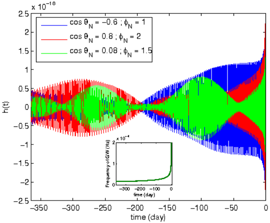

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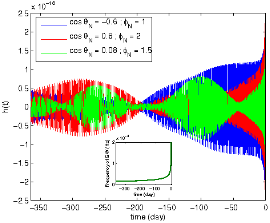

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$$h^{(\mathrm{RWF})}_{+,\times} = \frac{2M\eta}{D_L}(M\omega)^{2/3}\,H^{(0)}_{+,\times}$$

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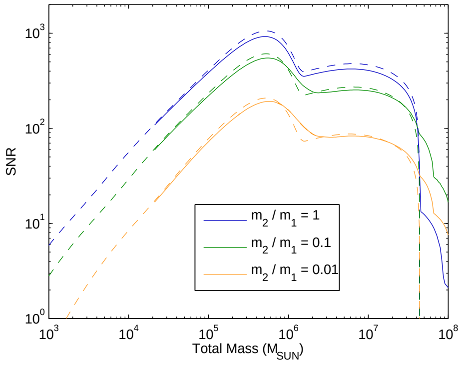
Impact on the SNR

We set . . .

- Sky position: $\cos\theta_N = -0.6$, $\phi_N = 1$ rad.
- Orientation: $\cos\theta_L = 0.2$, $\phi_L = 3$ rad.
- Redshift: $z = 1$ ($D_L = 6.64$ Gpc).



➢ Modeling the inspiral with FWF, extends LISA's reach to higher mass systems. [Arun et al. PRD 75, 124002 (2007)]

➢ The improvements we will find in the errors are not due to an increase of the SNR, but to the presence of higher harmonics (they disentangle the source parameters).

Reminder

☞ Frequency emission decreases with the mass of the BHs.

Parameter estimation

➢ Given two signals $g(t)$ and $h(t)$, one defines their inner product as:

$$(g,h) \equiv 2\int_0^\infty \frac{\tilde{g}^*(f)\tilde{h}(f) + \tilde{g}(f)\tilde{h}^*(f)}{S_n(f)}\,\mathrm{d}f$$

Optimal signal-to-noise ratio (SNR)

$$\rho^2 = (h,h) = 4\int_0^\infty \frac{|\tilde{h}(f)|^2}{S_n(f)}\,\mathrm{d}f$$

Expected root-mean-square errors in parameters (in the limit of high SNR)

➢ Fisher Information Matrix:

$$\Gamma_{ij} \equiv (\partial_i h, \partial_j h) = 2\int_0^\infty \frac{\partial_i\tilde{h}^*(f)\partial_j\tilde{h}(f) + \partial_i\tilde{h}(f)\partial_j\tilde{h}^*(f)}{S_n(f)}\,\mathrm{d}f \qquad \begin{cases} p(s|\theta) \propto e^{-\frac{(s-h(\theta),\,s-h(\theta))}{2}} \\ s = h_0 + n\ ;\ h(\theta) \end{cases}$$

➢ Variance-covariance matrix: $\Sigma^{jk} \equiv (\Gamma^{-1})^{jk} = \langle \Delta\theta^j \Delta\theta^k \rangle$

$$\sigma_k = \langle (\Delta\theta^k)^2 \rangle^{1/2} = \sqrt{\Sigma^{kk}} \qquad c^{jk} = \frac{\langle \Delta\theta^j \Delta\theta^k \rangle}{\sigma_j \sigma_k} = \frac{\Sigma^{jk}}{\sqrt{\Sigma^{jj}\Sigma^{kk}}}$$

More than one detector

$$\rho^{tot} = \sqrt{(\rho^I)^2 + (\rho^{II})^2 + \dots} \qquad \Gamma_{ij}^{tot} = \Gamma_{ij}^{I} + \Gamma_{ij}^{II} + \dots$$

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Analytical calculation of Fourier Transform

Stationary Phase Approximation

$$B(t) = A(t)\cos\left(\frac{j}{2}\Phi(t) + \varphi_0\right) \quad \longmapsto \quad \tilde{B}(\nu) = \left[\frac{A(t_0)}{2} e^{i(2\pi\nu t_0 - \frac{j}{2}\Phi - \varphi_0 - \frac{\pi}{4})}\sqrt{\frac{2}{j}\frac{1}{\frac{\mathrm{d}F}{\mathrm{d}t}}}\right]_{F=\frac{2}{j}\nu}$$

☞ We assume that $A(t)$ and $\left[\frac{j}{2}\Phi(t) + \varphi_0 - 2\pi\nu t\right] \equiv \xi(t)$ doesn't change so much with time

➢ Temporal evolution of the inspiral signal up to 2PN

$$h(t) = \sum_{j=1}^{6} h_j(t) = \sum_{j=1}^{6} \frac{\sqrt{3}}{2} 2M\eta\frac{1}{D_L}x^2 A_j \cos\left(\frac{j}{2}\Phi + \varphi_{p,j} + \varphi_{\mathrm{D}}\right)$$

$$\begin{cases} A_j = \mid \Upsilon(j) \mid \left[(\hat{u}_{+,j}F_{+} + \hat{u}_{\times,j}F_{\times})^2 + (\hat{w}_{\times,j}F_{\times} + \hat{w}_{+,j}F_{+})^2\right]^{1/2} \\ \varphi_{p,j} = \tan^{-1}\left[\frac{-(\hat{w}_{\times,j}F_{\times} + \hat{w}_{+,j}F_{+})}{(\hat{u}_{+,j}F_{+} + \hat{u}_{\times,j}F_{\times})}\right] \\ x \equiv (M\omega)^{1/3} \end{cases}$$

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➢ Analytical expression in the frequency domain (using the SPA):

$$\tilde{h}(\nu) = \sum_{j=1}^{6}\left[\frac{\tilde{h}_j}{2} e^{i\left[\frac{j}{2}(2\pi F t_c - \Phi) - \varphi_{p,j} - \varphi_{\mathrm{D}} - \frac{\pi}{4}\right]} \sqrt{\frac{2}{j}\frac{1}{\frac{dF}{dt}}}\right]_{F=\frac{2}{j}\nu}$$

where, $\tilde{h}_j \equiv \frac{\sqrt{3}}{2} 2M\eta \frac{1}{D_L} x^2 A_j$.

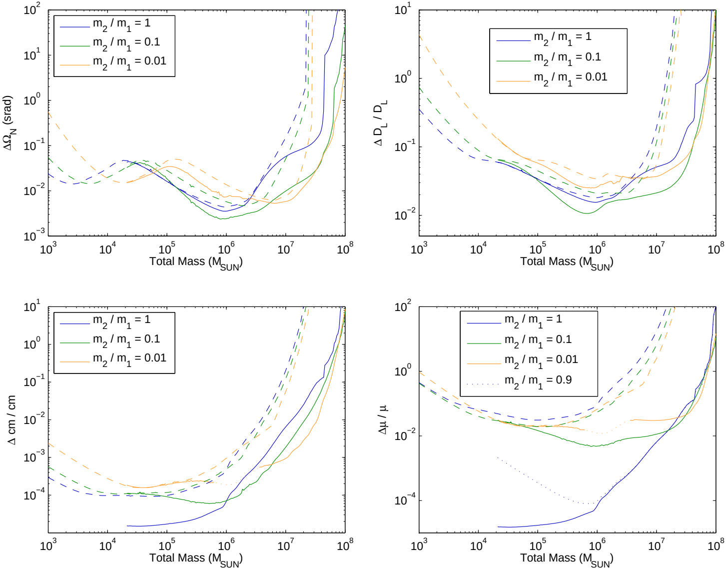
Main Assumptions

- We consider only the inspiral phase of the whole coalescence.
- We observe the source during the last year before the merger.
- We restrict ourselves to circular orbits.
- The waveform is approximated up to 2PN order (6 harmonics).
- We consider a flat Universe described by $H_0 = 71 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_m = 0.27$ and $\Omega_\Lambda = 0.73$.
- LISA noise is modeled as the sum of two contributions: instrumental noise and confusion noise (f is in Hz) ($f_{low \ cutoff} = 5 \times 10^{-5}$ Hz):

$$
\begin{aligned}
S_n^{\mathrm{inst}}(f) &= 6.12 \times 10^{-51} \ f^{-4} + 1.06 \times 10^{-40} + 6.12 \times 10^{-37} \ f^2 \ \mathrm{Hz}^{-1} \\
S_n^{\mathrm{GWD}}(f) &= 1.4 \times 10^{-44} \ f^{-7/3} \ \mathrm{Hz}^{-1} \\
S_n^{\mathrm{EWD}}(f) &= 2.8 \times 10^{-46} \ f^{-7/3} \ \mathrm{Hz}^{-1}
\end{aligned}
$$

- We are using the low frequency approximation to the LISA response: we see it as two Michelson.

Evolution with M, fixed the sky position



We set . . .

Sky position:
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Orientation:
$\cos\theta_L = 0.2$,
$\phi_L = 3$ rad.

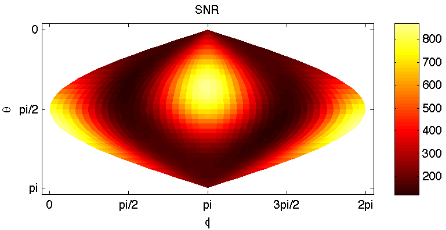
Redshift: $z = 1$
($D_L = 6.64$ Gpc).

Null spin parameters
($\beta = \sigma = 0$).

Sky distribution of SNR (working with FWF)

We set ...

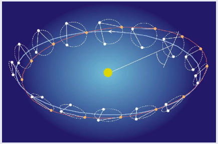
- Orientation: $\cos\theta_L = 0.2$, $\phi_L = 3$ rad.
- Redshift: $z = 1$ ($D_L = 6.64$ Gpc).
- Masses: $m_1 = m_2 = 10^7 M_\odot$.
- Signal model: FWF.



➢ The highest SNR is reached at ($\phi_N = 0^o; \theta_N = 120^o$) and ($\phi_N = 180^o; \theta_N = 60^o$)

We should have in mind that ...

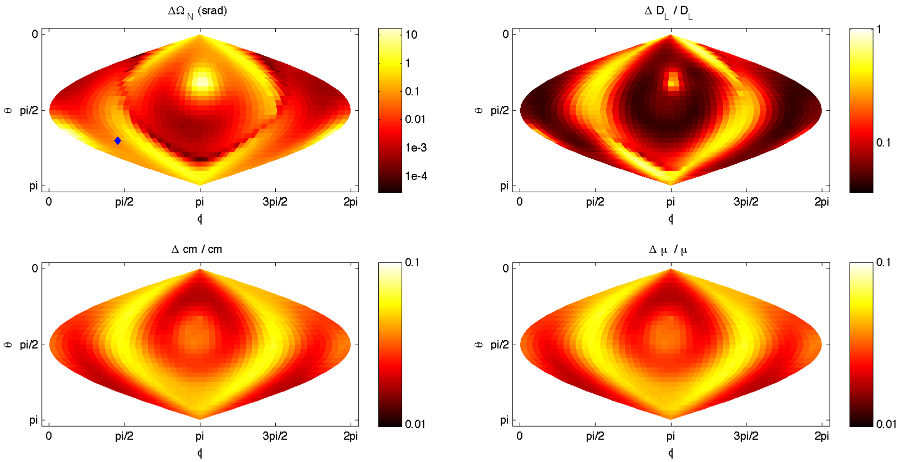
☞ LISA's orbit ($\phi_{\rm LISA}(t_c) = 0$):



☞ Detector's sensitivity is higher in the orthogonal direction to its plane.

☞ Massive systems accumulate most of the SNR in the last few days before merger.

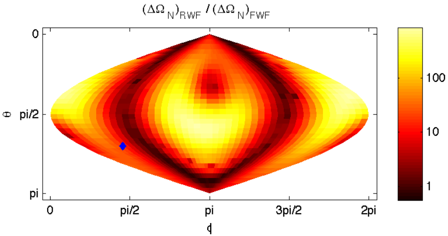
Sky distribution of errors (working with FWF)



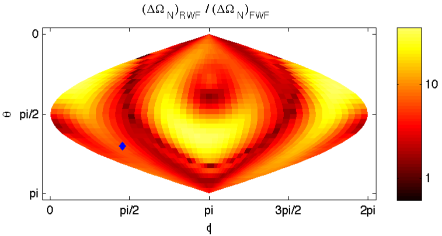
➢ It does not exist a direct correlation between SNR and errors.

➢ We find a strong variation of error with the sky position. Monte-Carlo simulations are required in order to extract general conclusions.

Gain in angular resolution



$m_1 = 10^7 M_\odot$; $m_2 = 10^7 M_\odot$



$m_1 = 10^7 M_\odot$; $m_2 = 10^6 M_\odot$

- Up to 3 orders of magnitude of gain in the angular resolution.
- Larger improvements when we are working in the range of very high masses.

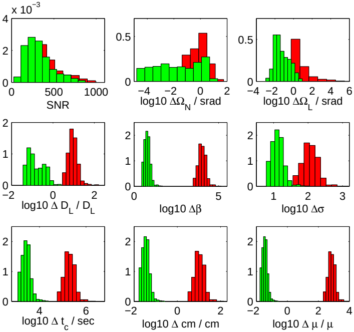
Extracting general conclusions. Exploring all sky with MCs

We set . . .

- Redshift: $z = 1$ ($D_L = 6.64$ Gpc).
- The pair of masses.

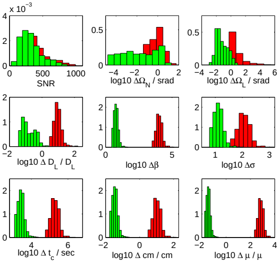
We do . . .

- 1000 MCs over random sky positions and orientations.
- Compare probability distributions of FWF and RWF.



➢ Plotting the FWF *vs* RWF results for the $m_1 = 10^7 M_\odot$; $m_2 = 10^7 M_\odot$ case.

Extracting general conclusions. Exploring all sky with MCs



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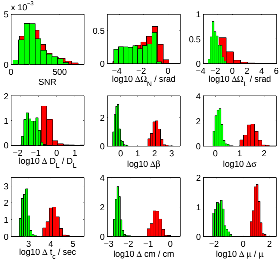
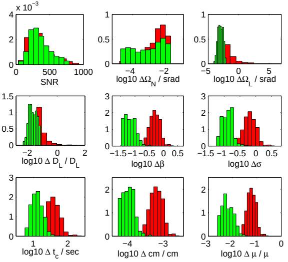


$m_1 = 10^7 M_\odot \;\; ; \;\; m_2 = 10^6 M_\odot$



$m_1 = 10^6 M_\odot \;\; ; \;\; m_2 = 10^5 M_\odot$

	SNR	$\Delta\Omega_N$ / srad	$\Delta D_L / D_L$	$\Delta\mathcal{M} / \mathcal{M}$	$\Delta\mu / \mu$
(1e7 – 1e7) $M_\odot$	0.87	25	62	300	20000
(1e7 – 1e6) $M_\odot$	0.92	7.3	6.7	87	600
(1e7 – 1e5) $M_\odot$	0.94	1.4	2.3	34	46
(1e6 – 1e6) $M_\odot$	0.90	2.1	1.8	9.1	1420
(1e6 – 1e5) $M_\odot$	1.0	2.9	2.7	6.1	9.6
(1e5 – 1e5) $M_\odot$	0.91	2.2	1.7	6.6	1760

⇐ Gain factors

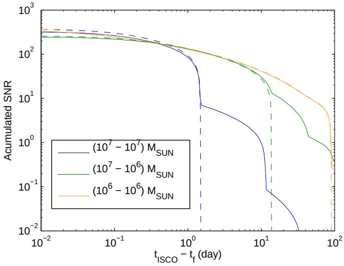
Pre-merger localization

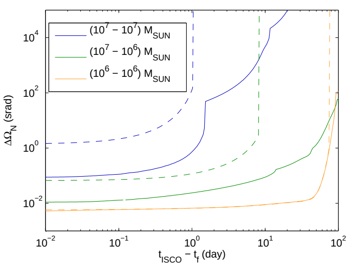
We set . . .

- Sky loc.: $\cos\theta_N = -0.6$, $\phi_N = 1$ rad
- Orientation: $\cos\theta_L = 0.2$, $\phi_L = 3$ rad
- Initial t_{obs}: 1 yr before ISCO
- The pair of masses.
- Redshift: $z = 1$.

We study . . .

☞ Evolution of SNR and angular resolution with the stopping time before ISCO.





- Important improvements in $10^7 M_\odot - 10^7 M_\odot$ and $10^7 M_\odot - 10^6 M_\odot$ systems, but no for lower mass binaries.
- We could create alerts sooner, gaining hours or even days, in massive systems.

Summary

☞ Modeling the inspiral waveform using FWF, instead of RWF:

- Extends the reach to higher mass systems (up to $M \simeq 10^8 M_\odot$).
- Importantly improves the errors in systems with $M \gtrsim 5 \times 10^6 M_\odot$.
- Specifically, for binary systems of $10^7 M_\odot - 10^7 M_\odot$ and $10^7 M_\odot - 10^6 M_\odot$ at $z = 1$, the angular resolution improves in average by factors of 25 and 7.3, resp. and the luminosity distance by 62 and 6.7, resp.
- Allows for early warnings of incoming mergers (for systems with a high total mass).

☞ Recent articles related with this topic:

- K. G. Arun et al, Phys. Rev. D76:104016 (2007).
- M. Trias and A. M. Sintes, arXiv:0707.4434 (to appear in Phys. Rev. D).