The cross-correlation search for periodic gravitational waves

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Outline

Existing CW searches

- General categories: Coherent or semi-coherent
- Coherent search implemented as the $\mathcal{F}$-statistic
- Semi-coherent searches:
  - Non-demodulated with short coherent segments (Stackslide, Powerflux,Hough)
  - Demodulated with longer coherent segments (Hierarchical search)
- The cross-correlation statistic – motivated by directed Stochastic searches and so far used for Sco X-1 search
- We want to tailor this further to CW searches

The Sco X-1 Radiometer search

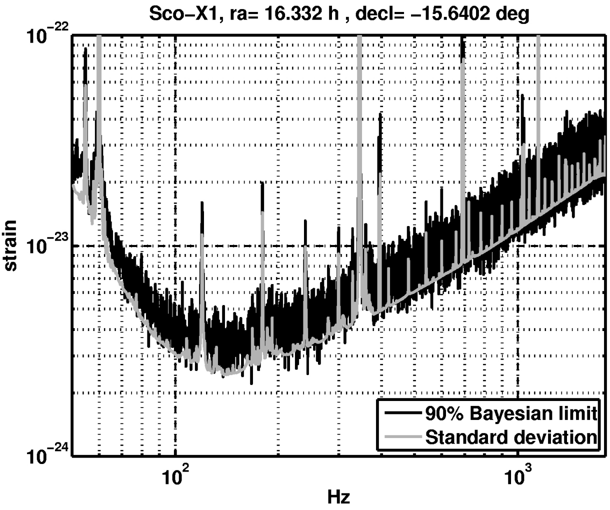
- L1 and H1 detectors see the same GW signal
- Introduce time delays in the two data streams
- Cross-correlate the data streams to extract the signal – aperture synthesis
- Details in `astro-ph/0703234`
- Basic statistic is the cross-correlation

$$Y_t = \int_{-\infty}^{\infty} \tilde{x}_1^\star(f) Q_t(f) \tilde{x}_2(f)$$

- Choice of optimal filter can be used to “point” the radiometer
- Upper limits are in 0.25 Hz bands

$$h^{rms} = 3.4 \times 10^{-24} \left(\frac{f}{200\text{Hz}}\right) \quad \text{for} \quad f > 200\,\text{Hz}$$

The Sco X-1 Radiometer search



The standard stochastic searches

Standard stochastic searches work under the following assumptions

- Statistical properties of signal are time independent
- Signal at different times are uncorrelated
- Polarizations are statistically independent

These assumptions do not hold for CW sources

- Signal has long term phase coherence
- Signal is not stationary (spindown and Doppler shift)
- The polarizations are not independent
- These effects are important for finer frequency resolutions

Notation

In the rest frame of the star, the signal is a slowly varying sinusoid with a quadrupole pattern:

$$h_+(\tau) = A_+ \cos\Phi(\tau) \qquad h_\times(\tau) = A_\times \sin\Phi(\tau)$$

$$A_+ = h_0 \frac{1+\cos^2\iota}{2} \qquad A_\times = h_0 \cos\iota$$

$$h_0 = \frac{16\pi^2 G}{c^4} \frac{I_{zz}\epsilon f_r^2}{d}$$

- ι: pulsar orientation w.r.t line of sight
- $\epsilon = (I_{xx} - I_{yy})/I_{zz}$: equatorial ellipticity
- f_r: rotation frequency
- d: distance to star

Notation

The phase in the rest frame of the star:

$$\Phi(\tau) = \Phi_0 + 2\pi \left[f(\tau - \tau_0) + \frac{1}{2}\dot{f}(\tau - \tau_0)^2 + \ldots \right]$$

Need to correct for the arrival times

- For an isolated pulsar:

$$\tau = t + \frac{\mathbf{r}_D \cdot \mathbf{n}}{c} + \text{relativistic corrections}$$

- For a pulsar in a binary system:

$$\tau = t + \frac{\mathbf{r}_D \cdot \mathbf{n}}{c} - \frac{\mathbf{r}_P \cdot \mathbf{n}}{c} + \text{relativistic corrections}$$

- $\mathbf{n}$: sky-position, $\mathbf{r}_D$: Detector in SSB frame, $\mathbf{r}_P$: Pulsar in binary frame

Notation

- The received signal is amplitude modulated due to the detector antenna pattern

$$h(t) = F_+(t)h_+(t) + F_\times(t)h_\times(t)$$

- ...and the frequency is Doppler modulated

$$f(t) - \hat{f}(t) = \hat{f}(t)\frac{\mathbf{v}(t)\cdot\mathbf{n}}{c}\,.$$

- Decomposition of $F_{+,\times}$:

$$F_+(t) = a(t)\cos 2\psi + b(t)\sin 2\psi\,,$$
$$F_\times(t) = b(t)\cos 2\psi - a(t)\sin 2\psi\,.$$

A general cross-correlation statistic

- Standard radiometer correlates coincident data from multiple IFOs
- For continuous waves we can do more
- We could correlate data from any two time segments
- The basic statistic is

$$\int_{T_1-\Delta T/2}^{T_1+\Delta T/2} dt_1 \int_{T_2-\Delta T/2}^{T_2+\Delta T/2} dt_2\, x_1(t_1)x_2(t_2)Q(t_1,t_2)\,,$$

where

$$x_1(t) \quad \text{is data for} \quad t \in [T_1-\Delta T/2, T_1+\Delta T/2]\,,$$
$$x_2(t) \quad \text{is data for} \quad t \in [T_2-\Delta T/2, T_2+\Delta T/2]\,.$$

- x_1 and x_2 could be data from same or different detectors

A general cross-correlation statistic

- Consider the time interval $[T - \Delta T/2, T + \Delta T/2]$
- We will assume signal power to be mostly in a single frequency bin over ΔT
- With this assumption, Fourier transform of the signal is

$$\tilde{h}(f) = e^{i\pi f \Delta T} \left[e^{i\Phi(T)} \frac{(F_+ A_+ - iF_\times A_\times)}{2} \delta_{\Delta T}(f - f(T)) \right. \\ \left. + e^{-i\Phi(T)} \frac{(F_+ A_+ + iF_\times A_\times)}{2} \delta_{\Delta T}(f + f(T)) \right]$$

- f is signal frequency at T, $\Phi(T)$ is phase at T, and $\delta_{\Delta T}$ is finite time approximation of δ-function

Optimal filter for a single SFT pair

- Let us start by assuming a time-invariant filter

$$\int dt \int dt' x_1(t) x_2(t') Q(t-t') = \int df \tilde{x}_1^\star(f) \tilde{x}_2(f) \tilde{Q}_{12}(f)$$

- Optimal filter is

$$\tilde{Q}_{12}(f) \propto \tilde{h}_1^\star(f) \tilde{h}_2(f)$$

- This is bad – if signal frequency is non-stationary, $\tilde{h}_1$ and $\tilde{h}_2$ may have very little overlap
- But easy to fix – shift frequency appropriately before correlating
- Leads to non time-invariant Q

$$Q(t,t') = e^{-i\pi(t+t')\delta f} Q(t-t')$$

Signal cross-correlation function

- After making the signals overlap, we can define

$$\tilde{h}_1^\star(f)\tilde{h}_2(f+\delta f) := h_0^2 \tilde{\mathcal{G}}_{12} \delta^2_{\Delta T}(f-f_1)$$

- Expression for $\tilde{\mathcal{G}}$:

$$\tilde{\mathcal{G}}_{12} = \frac{1}{4} e^{-i\Delta\Phi_{12}} \left\{ (F_{1+}F_{2+}\mathcal{A}_+^2 + F_{1\times}F_{2\times}\mathcal{A}_\times^2) - i(F_{I+}F_{J\times} - F_{I\times}F_{J+})\mathcal{A}_+\mathcal{A}_\times \right\}$$

- Here $\mathcal{A}_+$ and $\mathcal{A}_\times$ are the amplitudes with h_0 taken out

$$\mathcal{A}_+ = \frac{1+\cos^2\iota}{2}, \quad \mathcal{A}_\times = \cos\iota$$

- A useful result

$$\langle \tilde{\mathcal{G}}_{12} \rangle_{\cos\iota,\psi} = \frac{1}{10} e^{-i\Delta\Phi_{12}} (a_1 a_2 + b_1 b_2)$$

Analogous to overlap reduction function

The detection statistic

- Construct the raw cross-correlation between the I^{th} and J^{th} SFTs

$$\mathcal{Y}_{k,IJ} = \frac{\tilde{x}^\star_{k,I}\tilde{x}_{k',J}}{\Delta T^2}$$

- Assume bin indices are correctly shifted
- Denote SFT pair $\{IJ\}$ by single index α
- Our detection statistic is

$$\rho = \sum_\alpha (u_\alpha \mathcal{Y}_\alpha + u^*_\alpha \mathcal{Y}^*_\alpha)$$

- We want to choose weights u_α to maximize sensitivity

The optimal weights

- Choose false alarm and false dismissal rates α, β
- Smallest amplitude that can cross corresponding thresholds is

$$h_0 = \mathcal{S}\frac{||\mathbf{u}||}{\mathbf{u}\cdot\mathbf{H}}$$

where

$$\mathbf{x}\cdot\mathbf{y} := \sum_\alpha \mathrm{Re}\left[x_\alpha^* y_\alpha\right]\sigma_\alpha^2\,, \qquad H_\alpha = \tilde{\mathcal{G}}_\alpha^*/\sigma_\alpha^2$$

- The optimal weights which minimize h_0 are

$$u_\alpha \propto \frac{\tilde{\mathcal{G}}_\alpha^*}{\sigma_\alpha^2}$$

- This is closely analogous to powerflux choice of weights

Sensitivity

- The sensitivity is given by

$$h_0 = \frac{\mathcal{S}^{1/2}}{\sqrt{2}\langle|\tilde{\mathcal{G}}_\alpha|^2\rangle_\alpha^{1/4}} \frac{1}{N_{\mathrm{pairs}}^{1/4}} \sqrt{\frac{\left(S_n^{(1)} S_n^{(2)}\right)^{1/2}}{\Delta T}}$$

where

$$\sum_\alpha |\tilde{\mathcal{G}}_\alpha|^2 = N_{\mathrm{pairs}} \langle|\tilde{\mathcal{G}}_\alpha|^2\rangle_\alpha$$

and

$$\mathcal{S} = \mathrm{erfc}^{-1}(2\alpha) + \mathrm{erfc}^{-1}(2\beta)$$

- When we take all possible pairs, then $N_{\mathrm{pairs}} \sim N_{\mathrm{sft}}^2 \implies h_0 \propto T_{\mathrm{obs}}^{-1/2}$
- In this case ρ is a fully coherent statistic

Relation with the $\mathcal{F}$-statistic

- When we take all possible SFT pairs then ρ must be related with the $\mathcal{F}$-statistic
- Indeed, we can write $\mathcal{F}$ as

$$\mathcal{F} = \sum_{\alpha} (u_\alpha \mathcal{Y}_\alpha + u^*_\alpha \mathcal{Y}^*_\alpha)$$

- with the weights being

$$u_{IJ} \propto (Ab_I b_J + Ba_I a_J - C(a_I b_J + a_J b_I))\, e^{i\Delta\Phi_{IJ}}$$

where, as usual,

$$A = \int_0^{T_{\text{obs}}} a^2(t)\, dt\,, \qquad B = \int_0^{T_{\text{obs}}} b^2(t)\, dt$$

$$C = \int_0^{T_{\text{obs}}} a(t) b(t)\, dt$$

Relation with the $\mathcal{F}$-statistic

- If we take $A \approx B \gg C$ then these u_α corresponds to $\langle \tilde{\mathcal{G}}_{12} \rangle_{\cos\iota,\psi}$:
$$u_{IJ} \propto \langle \tilde{\mathcal{G}}_{IJ} \rangle_{\cos\iota,\psi}$$
- Extends very naturally to multi-detector $\mathcal{F}$-statistic
- Similarly, if we consider only self-correlations, ρ is identical to powerflux (can consider projections for linear and circular polarizations by choice of u_α)

Choosing SFT pairs

- Let us narrow down free choices by giving a criteria for choosing SFT pairs
- Choose a duration T_{max}
- The I^{th} and J^{th} SFTs are correlated if

$$|T_I - T_J| \leq T_{\mathrm{max}}$$

- $T_{\mathrm{max}} = T_{\mathrm{obs}} \implies$ full coherent search
- $T_{\mathrm{max}} = 0$ and distinct IFOs $\implies$ standard radiometer
- $T_{\mathrm{max}} = 0$ and self correlations $\implies$ powerflux
- Intermediate range $\implies$ Hierarchical semi-coherent search (but not exactly because here we also include correlations between stacks)

Conclusions

- Cross-correlation statistic considered for pulsar searches
- Provides general framework for existing CW techniques
- Method is very flexible: interpolates between full coherent, semi-coherent, hierarchical and standard cross-correlation methods
- Requires tuning based on computational cost and assumptions on signal model
- Important to work out parameter space metric